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A Mathematical Analysis for the Preservation of Forestry Biomass Using the Laplace Decomposition Method

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Abstract: In this work, a numerical analysis of a mathematical model for the preservation of forestry biomass is investigated. The model is divided into three compartments as density of forest biomass, density of wood based industries and density of synthetic industries. The Laplace Decomposition Method is used to obtain approximate solutions in the form of infinite series. Numerical justification is performed on the model parameter values with the aid of Maple 18 software to obtain the results. The behavior of the results obtained, is presented graphically. From the results, it was observed that the population of forest biomass increases exponentially as we increase the competitive effect of forest biomass c_1, on wood industries. It was also observed that the wood based industries will have no depleting effect on the forest biomass even when the competitive effect parameter of wood based industries c_2, on forest biomass was increased, and this was likened to increase awareness on synthetics as alternatives to wood, government control policies on deforestation, and an increase in prices of timber. It was also obvious from the result that as sufficient synthetic materials are supplied to the synthetic industries, the industries explode exponentially with time, and would serve as a good alternative to wood inpreserving the forestry biomass.

Keywords: Biomass, Infinite Series, Laplace Decomposition, Density

1. Introduction

In recent years, the depletion of forest, bio-diversity loss and changes in climate are closely linked to high rate of deforestation. This has made the environment not to be eco-friendly. One of the primary reasons for this deforestation is to harness timber products (wood) which serve diverse functions to human need. Wood is used for making furniture, houses, cooking, baskets, papers, etc. However, the negative impact of deforestation outweighs its benefits of preserving forestry resources for mankind. There is an urgent need to ensure the continuity and services rendered by forests to humanity, and to address this threatening problem of the universe, and reduce the excruciating effects of wood industry on forestry biomass, provision of synthetics would serve as suitable alternative to reduce the impact of wood industry on forest biomass. In addition, choosing a specific level of deforestation, increasing the density of forest resources through planting, and subjecting the wood based industries under control either by the action of government agencies or human awareness, could help in conserving our forest biomass. Hence, the task of this study is, therefore, to consider and analyze a mathematical model of forestry biomass preservation.

For the past decades, there has been so much interest in dynamical characteristics of population model. Forest biomass system happens to play an important role in population dynamics for the past decades, there has been so much interest in dynamical characteristics of population model. Forest biomass system happens to play an important role in population dynamics Agarwal et al [1]. The study of forests has been traced to have its starting point from the German scientific forestry for the concept of sustainability development during the eighteenth and nineteenth centuries Richard [2]. Thereafter, the study of forests and forestry preservation has gained tremendous attention in government, among individuals and researchers. A number of theoretical and scientific efforts have been made in literature, to explain the degradation of the forest ecosystem, its impact, as well as measures of conserving it.

Adekola and Mbalisi [3] considered community education in Nigeria as a practical community-based measure to environmental adult education, required to enlighten the rural inhabitants over the dangers of degradation of the forest ecosystem and acquaint them with the activities that lead to degradation of the forest ecosystem. The study further concludes that the community education of the rural inhabitants for forest conservation and preservation is key, irrespective of whether their activities contribute or not to the forest ecosystem degradation.

Dubrey and Freedom [4] worked on a mathematical model to check the growth and survival of resources biomass-dependents species in a forested habitat that is being depleted as a result of industrialization. They showed that the biomass density decreases, leading to the lowering of the density of species and its eventual extinction if this pressure continues unabated. However, it is established that if strategic effects are made to conserve the resource biomass, and also control the intensity of industrialization in the forested habitat, the survival of resource biomass-dependent species can be ensured.

Teru and Koya [5] investigated on a non-linear mathematical model to study the deforestation of forest resources due to lack of awareness about utility of the forest, as well as to promote forestry resources by afforestation. The formulated model was in form of ordinary differential equations with results showing that as the density of population together with population pressures intensifies, the cumulative density of forestry resources declines.

Muhammad, et al [6] proposed and analyzed a numerical solution of SEIR epidermic model of measles with non-integer time fractional derivatives by using Laplace Adomian Decomposition Me.

Chaudhary and Dhar [7] examined a mathematical model for forestry biomass with a maturation delay, such as pre-mature trees and mature trees, and two types of industrializations as wood based industries and synthetic industries.

Rachana [8] proposed and analyzed a non-linear mathematical model. He assumed that the rate of growth of wildlife population totally depends on forestry biomass. In a similar study, Jyotsna and Tandon [9] investigated a non-linear mathematical model to check the impact of mining activities and pollution on forest resources and wildlife populations. They applied numerical simulations and element of stability theory to analyze their model. Agyemang and Freedom [10] worked on the environmental model for the interaction of industry with two competing agricultural resources using a system of non-linear differential equation. They worked on the long term effects of each of these assets on each other. Chaudhary et al [11] proposed a mathematical model for conservation of forest biomass with wood based industries and synthetic industries. Criteria for local stability, instability, as well as global stability of non-negative equilibra were obtained. Similarly, Misra and Lata [12] proposed a mathematical model to analyze the depletion and conservation of forestry resources in the presence of industrialization. Their model showed that the increase in the carrying capacity of forest biomass due to technological efforts has destabilizing effect. They also support their analytical results with numerical simulations.

Chaudhary et al [13] considered an age structured forestry biomass classified as pre-mature and mature population stages, and industrialization as a state variable. Lata et al [14] worked on a mathematical model to check the effects of wood and non-wood based industries on depletion of forestry resources. Elizabeth and Victor [15] proposed and analyzed a non-linear mathematical model to study the deforestation due to human population and the effect it poses on farm fields. Rajinder [16] studied a two-dimensional mathematical model comprising of forest biomass and industry. Several other investigations have been carried out recently to study the interactions between the forestry population, population of wood based industries and density of synthetic industries, in relation to the depletion of forest biomass as a result of deforestation and conservative measures. However, most of the researchers and experimentalists have concentrated on studying the behavior of the mathematical model with stability analysis and numerical simulation analysis in the presence of the model parameter values. In this work, we have proposed a new method of approach - The Laplace Decomposition Method, to analyze the model and obtain an approximate solution in form of an infinite series. The Laplace Adomian Decomposition Method (LADM) is strongly and simply capable of solving the non-linear dynamical systems, and finding the numerical solution of the system of differential models as seen in Bazuaye [17], Bazuaye and Ezeora [18], Chasnov. [19], Hussain [20] and Bazuaye Bazuaye [21].

2. Materials and Methods

In this session, we adopt the Laplace Decomposition Method on mathematical model proposed by Chaudhary et al [11].

The formulation of the model is defined by the following basic assumptions as follows:

The forestry population grows logistically in the absence of wood based industries.

The rate of extinction of the forestry biomass is as a result of the presence of the wood based industries.

The forestry biomass promotes the growth of wood industries.

The synthetic industries grow, independent of the forest biomass density.

There are alternatives provided to industries for the preservation of forest biomass.

Competition between wood based industries and synthetic industries are according to demand from the market.

2.1. Mathematical Formulation

Following Chaudhary et al [11], we have considered the mathematical model for dynamical systems of differential equations. The model is divided into three sectors: The forest biomass density, density of wood based industries and density of synthetic industries.

$$\frac{dB}{dt} = \phi B\left(1 - \frac{B}{q}\right) - d_1 BW,$$

$$\frac{dW}{dt} = \alpha_1 BW - c_1 WS - d_2 W, \quad (1)$$

$$\frac{dS}{dt} = K - c_2 WS - d_3 S,$$

With non-negative initial population defined as:

$$B(0) = B_0 > 0, W(0) = W_0 > 0, S(0) = S_0 > 0 \quad (2)$$

Modification of (1) gives

$$\frac{dB}{dt} = \phi B - \frac{\phi B^2}{q} - d_1 B\,W \quad (3)$$

$$\frac{dW}{dt} = \alpha_1 BW - c_1 WS - d_2 W \quad (4)$$

$$\frac{dS}{dt} = K - c_2 WS - d_3 S \quad (5)$$

Where the parameters are defined as follows:

B(t) is the population of the forest biomass at any time t

W(t) is the population of wood based industries at any time t

S(t) is the population of synthetic industries at any time t

ϕ is the intrinsic growth rate at which forest population grows logistically without wood based industries

q is the carrying capacity

c_1 is the competitive effect of forest density on wood based industries

c_2 is the competitive effect of wood based industries on forest biomass

K is the amount of synthetic supplied to the synthetic industries

d_1 is the rate of depletion of the forest biomass

d_2 is the natural rate of depletion of the wood industries

d_3 is the natural rate of depletion of the synthetic industries

α_1 is the rate at which wood industries grows in the presence of forest biomass

2.2. Applications of Laplace Decomposition Method

Laplace transform is a mathematical operation used to convert a system of differential equations to a system of algebraic equation, Bazuaye and Ezeora [18]. It transforms a variable (such as x, or y, or z in space or at time t) to a parameter(s)- a 'constant' under certain conditions. It transforms one variable at a time. Applying Laplace transform on both sides of the model (3) to (5) above, we obtain the system of equations.

$$\mathcal{L}\left[\frac{dS(t)}{dt}\right] = \mathcal{L}[K - c_2 W(t)S(t) - d_3 S(t)] \quad (6)$$

Using the property of Laplace transform on the LHS of equation (6), we have

$$\mathcal{L}\left[\frac{dB(t)}{dt}\right] = sB(s) - B(0) = s\mathcal{L}[B(t)] - B(0)$$

$$\mathcal{L}\left[\frac{dW(t)}{dt}\right] = sW(s) - W(0) = s\mathcal{L}[W(t)] - W(0)$$

$$\mathcal{L}\left[\frac{dS(t)}{dt}\right] = sS(s) - s(0) = s\mathcal{L}[S(t)] - s(0)$$

Note that from

$$\mathcal{L}[B(t)] = B(s), \mathcal{L}[W(t)] = W(s) \text{ and } \mathcal{L}[S(t)] = S(s) \quad (7)$$

Substituting equation (7) into equation (6), we obtain

$$\begin{aligned} s\mathcal{L}[B(t)] - B(0) &= \mathcal{L}\left[\phi B(t) - \frac{\phi}{q}[B(t)]^2 - d_1 B(t)W(t)\right] \\ s\mathcal{L}[W(t)] - W(0) &= \mathcal{L}[\alpha_1 B(t)W(t) - c_1 W(t)S(t) - d_2 W(t)] \\ s\mathcal{L}[S(t)] - S(0) &= \mathcal{L}[K - c_2 W(t)S(t) - d_3 W(t)] \end{aligned} \quad (8)$$

which gives

$$\begin{aligned} s\mathcal{L}[B(t)] &= B(0) + \mathcal{L}\left[\phi B(t) - \frac{\phi}{q}[B(t)]^2 - d_1 B(t)W(t)\right] \\ s\mathcal{L}[W(t)] &= W(0) + \mathcal{L}[\alpha_1 B(t)W(t) - c_1 W(t)S(t) - d_2 W(t)] \\ s\mathcal{L}[S(t)] &= S(0) + \mathcal{L}[K - c_2 W(t)S(t) - d_3 S(t)] \end{aligned} \quad (9)$$

Dividing through by s

$$\mathcal{L}[B(t)] = \frac{B(0)}{s} + \frac{1}{s}\left\{\mathcal{L}\left[\phi B(t) - \frac{\phi}{q}[B(t)]^2 - d_1 B(t)W(t)\right]\right\}$$

$$\mathcal{L}[W(t)] = \frac{W(0)}{s} + \frac{1}{s}\{\mathcal{L}[\alpha_1 B(t)W(t) - c_1 W(t)S(t) - d_2 W(t)]\} \quad (10)$$

$$\mathcal{L}[S(t)] = \frac{S(0)}{s} + \frac{1}{s}\{\mathcal{L}[K - c_2 W(t)S(t) - d_3 S(t)]\}$$

From (10), we have

$$\mathcal{L}[B(t)] = \frac{B(0)}{S} + \frac{\phi}{s}\mathcal{L}[B(t)] - \frac{\phi}{sq}\mathcal{L}[B(t)]^2 - \frac{d_1}{s}\mathcal{L}[B(t)W(t)]$$

$$\mathcal{L}[W(t)] = \frac{W(0)}{S} + \frac{\alpha_1}{s}\mathcal{L}[B(t)W(t)] - \frac{c_1}{s}\mathcal{L}[W(t)S(t)] - \frac{d_2}{s}\mathcal{L}[W(t)] \quad (11)$$

$$\mathcal{L}[S(t)] = \frac{S(0)}{S} + \frac{K}{s} - \frac{c_2}{s}\mathcal{L}[w(t)S(t)] - \frac{d_3}{s}\mathcal{L}[S(t)]$$

It is assumed that the method gives the solutions $B(t), W(t)$ and $S(t)$ in form of infinite series given as $B(t) = \sum_{i=0}^{\infty} B_i(t)$

$$W(t) = \sum_{i=0}^{\infty} W_i(t) \quad (12)$$

$$S(t) = \sum_{i=0}^{\infty} S_i(t)$$

and by decomposition approach, the nonlinearity solutions B(t)W(t) and W(t)S(t) in equation (11) can be written as

$$B(t)W(t) = \sum_{i=0}^{\infty} A_i(t) \quad (13)$$

$$W(t)S(t) = \sum_{i=0}^{\infty} Z_i(t)$$

Where each A_i and Z_i is the Adomian Polynomials defined as

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[\sum_{j=0}^{i}\lambda^j B_j(t)\sum_{j=0}^{i}\lambda^j W_j(t)\right]\Big|_{\lambda=0} \quad (14)$$

$$Z_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[\sum_{j=0}^{i}\lambda^j W_j(t)\sum_{j=0}^{i}\lambda^j S_j(t)\right]\Big|_{\lambda=0} \quad (15)$$

From (14), for i= 0, j = 0

$$A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}[\lambda^0 B_0(t).\lambda^0 W_0(t)]\Big|_{\lambda=0}$$

$$A_0 = \frac{1}{1}[B_0(t).W_0(t)]|_{\lambda=0}$$

$$A_0 = B_0(t).W_0(t) \quad (16)$$

For i = 1, j = 0,1

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[\sum_{j=0}^{i}\lambda^j B_j(t)\sum_{j=0}^{i}\lambda^j W_j(t)\right]\Bigg|_{\lambda=0}$$

$$A_1 = \frac{1}{1!}\frac{d^1}{d\lambda^1}\left[\sum_{j=0}^{1}\lambda^j B_j(t)\sum_{j=0}^{1}\lambda^j W_j(t)\right]\Bigg|_{\lambda=0}$$

$$= \frac{1}{1}\frac{d}{d\lambda}\left[\left(\lambda^0 B_0(t) + \lambda^1 B_1(t)\right)\left(\lambda^0 W_0(t) + \lambda^1 W_l(t)\right)\right]\Big|_{\lambda=0}$$

$$= \frac{d}{d\lambda}\left[\left(B_0(t) + \lambda^1 B_1(t)\right)\left(W_0(t) + \lambda^1 W_1(t)\right)\right]\Big|_{\lambda=0}$$

Expanding the terms in the brackets

$$\frac{d}{d\lambda}\left[[B_0(t)W_0(t) + \lambda^1 B_0(t)W_1(t) + \lambda^1 B_1(t)W_0(t) + \lambda^2 B_1(t)W_1]\right]\Big|_{\lambda=0}$$

By differentiating with respect to λ

$$A_1 = [B_0(t)W_1(t) + B_1(t)\,W_0(t) + 2\lambda\, B_l(t)\,W_1(t)]|_{\lambda=0}$$

Substituting λ= 0

$$A_1 = B_0(t)\,W_1(t) + B_1(t)\,W_0(t) + 2[0]B_l(t)\,W_1(t)$$

Therefore,

$$A_1 = B_0(t)W_1(t) + B_1(t)W_0(t) \quad (17)$$

Also, for i = 2, j = 0, 1, 2
Then

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[\sum_{j=0}^{2}\lambda^j B_j(t)\sum_{j=0}^{2}\lambda^j W_j(t)\right]\Bigg|_{\lambda=0}$$

$$A_2 = \frac{l}{2}\frac{d^2}{d\lambda^2}\left[\left(\lambda^0 B_0(t) + \lambda^1 B_1(t) + \lambda^2 B_2(t)\right)\left(\lambda^0 W_0(t) + \lambda^1 W_1(t) + \lambda^2 W_2(t)\right)\right]\Big|_{\lambda=0}$$

$$= \frac{1}{2}\frac{d^2}{d\lambda^2}\left[\left(B_0(t) + \lambda^1 B_1(t) + \lambda^2 B_2(t)\right)\left(W_0(t) + \lambda^1 W_1(t) + \lambda^2 W_2(t)\right)\right]\Big|_{\lambda=0}$$

Expanding the terms in the brackets yields

$$A_2 = \frac{1}{2}\frac{d^2}{d\lambda^2}[B_0(t)W_0(t) + \lambda^1 B_0(t) W_1(t) + \lambda^2 B_0(t)W_2(t)$$
$$+ \lambda^1 B_1(t)W_0(t) + \lambda^2 B_1(t)W_1(t) + \lambda^3 B_1(t)W_2(t) + \lambda^2 B_2(t)W_0(t) + \lambda^3 B_2(t)W_1(t) + \lambda^4 B_2(t)W_2(t)]\big|_{\lambda=0}$$

First derivative with respect to λ gives

$$A_2 = \frac{1}{2}\frac{d}{d\lambda}[B_0(t)W_1(t) + 2\lambda B_0(t)W_2(t) + B_1(t)W_0(t) + 2\lambda B_1(t)W_1(t) + 3\lambda^2 B_1(t)W_2(t) + 2\lambda B_2(t)W_0(t) + 3\lambda^2 B_2(t)W_1(t) + 4\lambda^3 B_2(t)W_2(t)]\big|_{\lambda=0}$$

Second derivative with respect to λ gives

$$A_2 = \frac{1}{2}[2B_0(t)W_2(t) + 2B_1(t)W_1(t) + 6\lambda B_1(t)W_2(t) + 2B_2(t)W_0(t) + 6\lambda B_2(t)W_1(t) + 12\lambda^2 B_2(t)W_2(t)]\big|_{\lambda=0}$$

Substituting λ= 0,

$$A_2 = \frac{1}{2}[2B_0(t)W_2(t) + 2B_1(t)W_1(t) + 6[0]B_1(t)W_2(t) + 2B_2(t)W_0(t) + 6[0]B_2(t)W_1(t) + 12[0]^2 B_2(t)W_2(t)$$

$$= \frac{1}{2}[2B_0(t)W_2(t) + 2B_1(t)W_1(t) + 2\, B_2(t)W_0(t)]$$

Hence,

$$A_2 = B_0(t)W_2(t) + B_1(t)W_1(t) + B_2(t)W_0(t) \tag{18}$$

Similarly, for i = 3, j = 0,1,2,3

$$A_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}\left[\sum_{j=0}^{3}\lambda^j B_j(t)\sum_{j=0}^{3}\lambda^j W_j(t)\right]\Bigg|_{\lambda=0}$$

$$A_3 = \frac{1}{6}\frac{d^3}{d\lambda^3}\left[\left(\lambda^0 B_0(t) + \lambda^1 B_1(t) + \lambda^2 B_2(t) + \lambda^3 B_3(t)\right)\left(\lambda^0 W_0(t) + \lambda^1 W_l(t) + \lambda^2 W_2(t) + \lambda^3 W_3(t)\right)\right]\Big|_{\lambda=0}$$
$$= \frac{1}{6}\frac{d^3}{d\lambda^3}\left[\left(B_0(t) + \lambda^1 B_1(t) + \lambda^2 B_2(t) + \lambda^3 B_3(t)\right)\left(W_0(t) + \lambda^1 W_l(t) + \lambda^2 W_2(t) + \lambda^3 W_3(t)\right)\right]\Big|_{\lambda=0}$$

Expanding the terms in bracket yields

$$A_3 = \frac{1}{6}\frac{d^3}{d\lambda^3}[B_0(t)W_0(t) + \lambda^1 B_0(t)W_1(t) + \lambda^2 B_0(t)W_2(t) + \lambda^3 B_0(t)W_3(t) + \lambda^1 B_1(t)W_0(t) + \lambda^2 B_1(t)W_l(t) + \lambda^3 B_1(t)W_2(t) + \lambda^4 B_1(t)W_3(t) + \lambda^2 B_2(t)W_0(t) + \lambda^3 B_2(t)W_1(t) + \lambda^4 B_2(t)W_2(t) + \lambda^5 B_2(t)W_3(t) + \lambda^3 B_3(t)W_0(t) + \lambda^4 B_3(t)W_1(t) + \lambda^5 B_3(t)W_2(t) + \lambda^6 B_3(t)W_3(t)]\big|_{\lambda=0}$$

First derivative with respect to λ gives

$$A_3 = \frac{1}{6}\frac{d^2}{d\lambda^2}[B_0(t)W_1(t) + 2\lambda B_0(t)W_2(t) + 3\lambda^2 B_0(t)W_3(t) + B_1(t)W_0(t) + 2\lambda B_1(t)W_l(t) + 3\lambda^2 B_1(t)W_2(t) + 4\lambda^3 B_1(t)W_3(t) + 2\lambda B_2(t)W_0(t) + 3\lambda^2 B_2(t)W_1(t) + 4\lambda^3 B_2(t)W_2(t) + 5\lambda^4 B_2(t)W_2(t) + 3\lambda^2 B_3(t)W_0(t) + 4\lambda^3 B_3(t)W_1(t) + 5\lambda^4 B_3(t)W_2(t) + 6\lambda^5 B_3(t)W_3(t)]|_{\lambda=0}$$

Second derivative with respect λ gives

$$A_3 = \frac{1}{6}\frac{d}{d\lambda}[2B_0(t)W_2(t) + 6\lambda B_0(t)W_3(t) + 2B_1(t)W_1(t) + 6\lambda B_1(t)W_2(t) + 12\lambda^2 B_1(t)W_3(t) + 2B_2(t)W_0(t) + 6\lambda B_2(t)W_1(t) + 12\lambda^2 B_2(t)W_2(t) + 20\lambda^3 B_2(t)W_2(t) + 6\lambda B_3(t)W_0(t) + 12\lambda^2 B_3(t)W_1(t) + 20\lambda^3 B_3(t)W_2(t) + 30\lambda^4 B_3(t)W_3(t)]|_{\lambda=0}$$

Third derivative with respect to λ gives

$$A_3 = \frac{1}{6}[6B_0(t)W_3(t) + 6B_1(t)W_2(t) + 24\lambda B_1(t)W_3(t) + 6B_2(t)W_1(t) + 24\lambda B_2(t)W_2(t) + 60\lambda^2 B_2(t)W_2(t) + 6B_3(t)W_0(t) + 24\lambda B_3(t)W_1(t) + 60\lambda^2 B_3(t)W_2(t) + 120\lambda^3 B_3(t)W_3(t)]|_{\lambda=0}$$

Substituting $\lambda = 0$ yields

$$A_3 = \frac{1}{6}[6B_0(t)W_3(t) + 6B_1(t)W_2(t) + 24[0]B_1(t)W_3(t) + 6B_2(t)W_1(t) + 24[0]B_2(t)W_2(t) + 60[0]^2 B_2(t)W_2(t) + 6B_3(t)W_0(t) + 24[0]B_3(t)W_1(t) + 60[0]^2 B_3(t)W_2(t) + 120[0]^3 B_3(t)W_3(t)]$$

$$= \frac{1}{6}[6B_0(t)W_3(t) + 6B_1(t)W_2(t) + 6B_2(t)W_1(t) + 6B_3(t)W_0(t)]$$

Therefore,

$$A_3 = B_0(t)W_3(t) + B_1(t)W_2(t) + B_2(t)W_1(t) + B_3(t)W_0(t) \tag{19}$$

Hence,

$$A_0 = B_0(t)W_0(t)$$

$$A_1 = B_0(t)W_1(t) + B_1(t)W_0(t)$$

$$A_2 = B_0(t)W_2(t) + B_1(t)W_1(t) + B_2(t)W_0(t)$$

$$A_3 = B_0(t)W_3(t) + B_1(t)W_2(t) + B_2(t)W_1(t) + B_3(t)W_0(t)$$

Using similar fashion for equation (15), that is,

$$Z_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[\sum_{j=0}^{i}\lambda^j W_j(t)\sum_{j=0}^{i}\lambda^j S_j(t)\right]\Big|_{\lambda=0} \tag{20}$$

For i= 0, j = 0

$$Z_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}[\lambda^0 W_0(t).\lambda^0 S_0(t)]|_{\lambda=0}$$

$$Z_0 = \frac{1}{1}[W_0(t).S_0(t)]|_{\lambda=0}$$

$$Z_0 = W_0(t).S_0(t) \tag{21}$$

For i= 0, j = 0,1

$$Z_1 = \frac{1}{1!}\frac{d^1}{d\lambda^1}\left[\sum_{j=0}^{1}\lambda^j W_j(t)\sum_{j=0}^{1}\lambda^j S_j(t)\right]\Bigg|_{\lambda=0}$$

$$= \frac{1}{1}\frac{d}{d\lambda}\left[\left(\lambda^0 W_0(t) + \lambda^1 W_1(t)\right)\left(\lambda^0 S_0(t) + \lambda^1 S_1(t)\right)\right]\Big|_{\lambda=0}$$

$$= \frac{d}{d\lambda}\left[\left(W_0(t) + \lambda^1 W(t)\right)\left(S_0(t) + \lambda^1 S_1(t)\right)\right]\Big|_{\lambda=0}$$

Expanding the terms in the brackets

$$\frac{d}{d\lambda}[[W_0(t)S_0(t) + \lambda^1 W_0(t)S_1(t) + \lambda^1 W_1(t)S_0(t) + \lambda^2 W_1(t)S_1]]\Big|_{\lambda=0}$$

By differentiating with respect to λ

$$Z_1 = [W_0(t)S_1(t) + W_1 S_0(t) + 2\lambda W_1(t)S_1(t)]|_{\lambda=0}$$

Substituting $\lambda = 0$

$$Z_1 = W_0(t)\,S_1(t) + W_1(t)S_0(t) + 2[0]W_l(t)S_1(t)$$

Therefore,

$$Z_1 = W_0(t)\,S_1(t) + W_1(t)S_0(t) \tag{22}$$

Also, for i = 2, j = 0, 1, 2
Then

$$Z_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[\sum_{j=0}^{2}\lambda^j W_j(t)\sum_{j=0}^{2}\lambda^j S_j(t)\right]\Bigg|_{\lambda=0}$$

$$= \frac{1}{2}\frac{d^2}{d\lambda^2}\left[\left(\lambda^0 W_0(t) + \lambda^1 W_1(t) + \lambda^2 W_2(t)\right)\left(\lambda^0 S_0(t) + \lambda^1 S_1(t) + \lambda^2 S_2(t)\right)\right]\Big|_{\lambda=0}$$

$$= \frac{1}{2}\frac{d^2}{d\lambda^2}\left[\left(W_0(t) + \lambda^1 W_1(t) + \lambda^2 W_2(t)\right)\left(S_0(t) + \lambda^1 S_1(t) + \lambda^2 S_2(t)\right)\right]\Big|_{\lambda=0}$$

Expanding the terms in the brackets yields

$$Z_2 = \frac{1}{2}\frac{d^2}{d\lambda^2}[W_0(t)S_0(t) + \lambda^1 W_0(t)S_1(t) + \lambda^2 W_0(t)S(t)$$
$$+ \lambda^1 W_1(t)S_0(t) + \lambda^2 W_1(t)S_1(t) + \lambda^3 W_1(t)S_2(t) + \lambda^2 W_2(t)S_0(t) + \lambda^3 W_2(t)S_1(t) + \lambda^4 W_2(t)S_2(t)]|_{\lambda=0}$$

First derivative with respect to λ gives

$$Z_2 = \frac{1}{2}\frac{d}{d\lambda}[W_0(t)S_1(t) + 2\lambda W_0(t)S_2(t) + W_1(t)S_0(t) + 2\lambda W_1(t)S_1(t) + 3\lambda^2 W_1(t)S_2(t) + 2\lambda W_2(t)S_0(t) + 3\lambda^2 W_2(t)S_1(t) + 4\lambda^3 W_2(t)S_2(t)]|_{\lambda=0}$$

Second derivative with respect to λ gives

$$Z_2 = \frac{1}{2}[2W_0(t)S_2(t) + 2W_1(t)S_1(t) + 6\lambda W_1(t)S_2(t) + 2W_2(t)S_0(t) + 6\lambda W_2(t)S_1(t) + 12\lambda^2 W_2(t)S_2(t)]|_{\lambda=0}$$

Substituting λ = 0,

$$Z_2 = \frac{1}{2}[2W_0(t)S_2(t) + 2W_1(t)S_1(t) + 6[0]W_1(t)S_2(t) + 2W_2(t)S_0(t) + 6[0]W_2(t)S_1(t) + 12[0]^2 W_2(t)S_2(t)$$

$$= \frac{1}{2}[2W_0(t)S_2(t) + 2W_1(t)S_1(t) + 2W_2(t)S_0(t)]$$

Hence,

$$Z_2 = W_0(t)S_2(t) + W_1(t)S_1(t) + W_2(t)S_0(t) \tag{23}$$

Similarly for i = 3, j = 0,1,2,3

$$Z_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}\left[\sum_{j=0}^{3}\lambda^j W_j(t)\sum_{j=0}^{3}\lambda^j S_j(t)\right]\Big|_{\lambda=0}$$

$$Z_3 = \frac{1}{6}\frac{d^3}{d\lambda^3}\left[\left(\lambda^0 W_0(t) + \lambda^1 W_1(t) + \lambda^2 W_2(t) + \lambda^3 W_3(t)\right)\left(\lambda^0 S_0(t) + \lambda^1 S_l(t) + \lambda^2 S_2(t) + \lambda^3 S_3(t)\right)\right]\Big|_{\lambda=0}$$

$$= \frac{1}{6}\frac{d^3}{d\lambda^3}\left[\left(W_0(t) + \lambda^1 W_1(t) + \lambda^2 W_2(t) + \lambda^3 W_3(t)\right)\left(S_0(t) + \lambda^1 S_l(t) + \lambda^2 S_2(t) + \lambda^3 S_3(t)\right)\right]\Big|_{\lambda=0}$$

Expanding the terms in bracket yields

$$Z_3 = \frac{1}{6}\frac{d^3}{d\lambda^3}[W_0(t)S_0(t) + \lambda^1 W_0(t)S_1(t) + \lambda^2 W_0(t)S_2(t) + \lambda^3 W_0(t)S_3(t) + \lambda^1 W_1(t)S_0(t) + \lambda^2 W_1(t)S_l(t) + \lambda^3 W_1(t)S_2(t) + \lambda^4 W_1(t)S_3(t) + \lambda^2 W_2(t)S_0(t) + \lambda^3 W_2(t)S_1(t) + \lambda^4 W_2(t)S_2(t) + \lambda^5 W_2(t)S_3(t) + \lambda^3 W_3(t)S_0(t) + \lambda^4 W_3(t)S_1(t) + \lambda^5 W_3(t)S_2(t) + \lambda^6 W_3(t)S_3(t)]|_{\lambda=0}$$

First derivative with respect to λ gives

$$Z_3=\frac{1}{6}\frac{d^2}{d\lambda^2}[W_0(t)S_1(t)+2\lambda W_0(t)S_2(t)+3\lambda^2W_0(t)S_3(t)+W_1(t)S_0(t)+2\lambda W_1(t)S_1(t)+3\lambda^2W_1(t)S_2(t)+4\lambda^3W_1(t)S_3(t)+2\lambda W_2(t)S_0(t)+3\lambda^2W_2(t)S_1(t)+4\lambda^3W_2(t)S_2(t)+5\lambda^4W_2(t)S_2(t)+3\lambda^2W_3(t)S_0(t)+4\lambda^3W_3(t)S_1(t)+5\lambda^4W_3(t)S_2(t)+6\lambda^5W_3(t)S_3(t)]|_{\lambda=0}$$

Second derivative with respect λ gives

$$Z_3=\frac{1}{6}\frac{d}{d\lambda}[2W_0(t)S_2(t)+6\lambda W_0(t)S_3(t)+2W_1(t)S_1(t)+6\lambda W_1(t)S_2(t)+12\lambda^2W_1(t)S_3(t)+2W_2(t)S_0(t)+6\lambda W_2(t)S_1(t)+12\lambda^2W_2(t)S_2(t)+20\lambda^3W_2(t)S_2(t)+6\lambda W_3(t)S_0(t)+12\lambda^2W_3(t)S_1(t)+20\lambda^3W_3(t)S_2(t)+30\lambda^4W_3(t)S_3(t)]|_{\lambda=0}$$

Third derivative with respect to λ gives

$$Z_3=\frac{1}{6}[6W_0(t)S_3(t)+6W_1(t)S_2(t)+24\lambda W_1(t)S_3(t)+6W_2(t)S_1(t)+24\lambda W_2(t)S_2(t)+60\lambda^2W_2(t)S_2(t)+6W_3(t)S_0(t)+24\lambda W_3(t)S_1(t)+60\lambda^2W_3(t)S_2(t)+120\lambda^3W_3(t)S_3(t)]|_{\lambda=0}$$

Substituting $\lambda=0$ yields

$$Z_3=\frac{1}{6}[6W_0(t)S_3(t)+6W_1(t)S_2(t)+24[0]W_1(t)S_3(t)+6W_2(t)S_1(t)+24[0]W_2(t)S_2(t)+60[0]^2W_2(t)S_2(t)+6W_3(t)S_0(t)+24[0]W_3(t)S_1(t)+60[0]^2W_3(t)S_2(t)+120[0]^3W_3(t)S_3(t)]$$

$$=\frac{1}{6}[6W_0(t)S_3(t)+6W_1(t)S_2(t)+6W_2(t)S_1(t)+6W_3(t)S_0(t)]$$

Therefore,

$$Z_3=W_0(t)S_3(t)+W_1(t)S_2(t)+W_2(t)S_1(t)+W_3(t)S_0(t) \tag{24}$$

Hence,

$$\begin{aligned} Z_0 &= W_0(t)S_0(t)\\ Z_1 &= W_0(t)S_1(t)+W_1(t)S_0(t)\\ Z_2 &= W_0(t)S_2(t)+W_1(t)S_1(t)+W_2(t)S_0(t)\\ Z_3 &= W_0(t)S_3(t)+W_1(t)S_2(t)+W_2(t)S_1(t)+W_3(t)S_0(t) \end{aligned} \tag{25}$$

Next, we substitute equations (12) and (13) into equation (11), and we have

$$\begin{aligned} \mathcal{L}[\textstyle\sum_{i=0}^{\infty}B_i(t)] &= \frac{B(0)}{s}+\frac{\phi}{s}\mathcal{L}[\textstyle\sum_{i=0}^{\infty}B_i(t)]-\frac{\phi}{sq}\mathcal{L}[\textstyle\sum_{i=0}^{\infty}B_i(t)]^2-\frac{d_1}{s}\mathcal{L}[\textstyle\sum_{i=0}^{\infty}A_i(t)]\\ \mathcal{L}[\textstyle\sum_{i=0}^{\infty}W_i(t)] &= \frac{W(0)}{s}+\frac{\alpha_1}{s}\mathcal{L}[\textstyle\sum_{i=0}^{\infty}A_i(t)]-\frac{c_1}{s}\mathcal{L}[\textstyle\sum_{i=0}^{\infty}Z_i(t)]-\frac{d_2}{s}\mathcal{L}[\textstyle\sum_{i=0}^{\infty}W_i(t)]\\ \mathcal{L}[\textstyle\sum_{i=0}^{\infty}S_i(t)] &= \frac{S(0)}{s}+\frac{K}{s}-\frac{c_2}{s}\mathcal{L}[\textstyle\sum_{i=0}^{\infty}Z_i(t)]-\frac{d_3}{s}\mathcal{L}[\textstyle\sum_{i=0}^{\infty}S_i(t)] \end{aligned} \tag{26}$$

On simplification of equation (26) and using the initial conditions in (2), we obtain

$$\begin{aligned} \mathcal{L}\left[\sum_{i=0}^{\infty}B_i(t)\right] &= \frac{n_1}{s}+\frac{\phi}{s}\mathcal{L}\left[\sum_{i=0}^{\infty}B_i(t)\right]-\frac{\phi}{sq}\mathcal{L}\left[\sum_{i=0}^{\infty}B_i(t)\right]^2-\frac{d_1}{s}\mathcal{L}\left[\sum_{i=0}^{\infty}A_i(t)\right]\\ \mathcal{L}[\textstyle\sum_{i=0}^{\infty}W_i(t)] &= \frac{n_2}{s}+\frac{\alpha_1}{s}\mathcal{L}[\textstyle\sum_{i=0}^{\infty}A_i(t)]-\frac{c_1}{s}\mathcal{L}[\textstyle\sum_{i=0}^{\infty}Z_i(t)]-\frac{d_2}{s}\mathcal{L}[\textstyle\sum_{i=0}^{\infty}W_i(t)]\\ \mathcal{L}\left[\sum_{i=0}^{\infty}S_i(t)\right] &= \frac{n_3}{s}+\frac{K}{s}-\frac{c_2}{s}\mathcal{L}\left[\sum_{i=0}^{\infty}Z_i(t)\right]-\frac{d_3}{s}\mathcal{L}\left[\sum_{i=0}^{\infty}S_i(t)\right] \end{aligned} \tag{27}$$

Comparing the two sides of each equation in (27) results in the following iterative algorithm

$$\mathcal{L}[B_0]=\frac{n_1}{s}$$

$$\mathcal{L}[B_1] = \frac{\phi}{s^\mu}\mathcal{L}[B_0] - \frac{\phi}{s^\mu q}\mathcal{L}[B_0]^2 - \frac{d_1}{s^\mu}\mathcal{L}[A_0]$$

$$\mathcal{L}[B_2] = \frac{\phi}{s^\mu}\mathcal{L}[B_1] - \frac{\phi}{s^\mu q}\mathcal{L}[B_1]^2 - \frac{d_1}{s^\mu}\mathcal{L}[A_1] \tag{28}$$

$$\mathcal{L}[B_3] = \frac{\phi}{s^\mu}\mathcal{L}[B_2] - \frac{\phi}{s^\mu q}\mathcal{L}[B_2]^2 - \frac{d_1}{s^\mu}\mathcal{L}[A_2] \vdots$$

$$\mathcal{L}[B_{k+1}] = \frac{\phi}{s^\mu}\mathcal{L}[B_k] - \frac{\phi}{s^\mu q}\mathcal{L}[B_k]^2 - \frac{d_1}{s^\mu}\mathcal{L}[A_k]$$

Similarly,

$$\mathcal{L}[W_0] = \frac{n_2}{s}$$

$$\mathcal{L}[W_1] = \frac{\alpha_1}{s^\mu}\mathcal{L}[A_0] - \frac{c_1}{s^\mu}\mathcal{L}[Z_0] - \frac{d_2}{s^\mu}\mathcal{L}[W_0]$$

$$\mathcal{L}[W_2] = \frac{\alpha_1}{s^\mu}\mathcal{L}[A_1] - \frac{c_1}{s^\mu}\mathcal{L}[Z_1] - \frac{d_2}{s^\mu}\mathcal{L}[W_1] \tag{29}$$

$$\mathcal{L}[W_3] = \frac{\alpha_1}{s^\mu}\mathcal{L}[A_2] - \frac{c_1}{s^\mu}\mathcal{L}[Z_2] - \frac{d_2}{s^\mu}\mathcal{L}[W_2] \vdots$$

$$\mathcal{L}[W_{k+1}] = \frac{\alpha_1}{s^\mu}\mathcal{L}[A_k] - \frac{c_1}{s^\mu q}\mathcal{L}[Z_k] - \frac{d_2}{s^\mu}\mathcal{L}[W_k]$$

Also,

$$\mathcal{L}[S_0] = \frac{n_3}{s}$$

$$\mathcal{L}[S_1] = \frac{K}{s^\mu} - \frac{c_2}{s^\mu}\mathcal{L}[Z_0] - \frac{d_3}{s^\mu}\mathcal{L}[S_0]$$

$$\mathcal{L}[S_2] = \frac{K}{s^\mu} - \frac{c_2}{s^\mu}\mathcal{L}[Z_1] - \frac{d_3}{s^\mu}\mathcal{L}[S_1] \tag{30}$$

$$\mathcal{L}[S_3] = \frac{K}{s^\mu} - \frac{c_2}{s^\mu}\mathcal{L}[Z_2] - \frac{d_3}{s^\mu}\mathcal{L}[S_2]$$

$$\vdots$$

$$\mathcal{L}[S_{k+1}] = \frac{K}{s^\mu} - \frac{c_2}{s^\mu}\mathcal{L}[Z_k] - \frac{d_3}{s^\mu}\mathcal{L}[S_k]$$

At this point, we are now to analyze the mathematical behavior of the solutions B(t), W(t) and S(t). By applying the inverse Laplace transform to both sides of equations (28), (29) and (30), we get the values of B_0, W_0 and S_0 and use for further process. We substitute the values of B_0, W_0, S_0, A_0, and Z_0 into the individual equation to get the values of B_1, W_1, and S_1. Also, we find the remaining terms $B_2, B_3, B_4, \ldots$ $W_2, W_3, W_4, \ldots$ and $S_2, S_3, S_4, \ldots$ in the same manner.

From equation (28),

$$\mathcal{L}^{-1}\mathcal{L}[B_0] = \mathcal{L}^{-1}\left[\frac{n_1}{s}\right]$$

$$B_0 = n_1\mathcal{L}^{-1}\left[\frac{1}{s}\right] = n_1.1 = n_1$$

Therefore,

$$\Rightarrow B_0 = n_1$$

Also, from (29)

$$\mathcal{L}^{-1}\mathcal{L}[W_0] = \mathcal{L}^{-1}\left[\frac{n_2}{s}\right]$$

$$W_0 = n_2\mathcal{L}^{-1}\left[\frac{1}{s}\right] = n_2.1 = n_2$$

Therefore,

$$\Rightarrow W_0 = n_2$$

Similarly, from (30)

$$\mathcal{L}^{-1}\mathcal{L}[S_0] = \mathcal{L}^{-1}\left[\frac{n_3}{s}\right]$$

$$S_0 = n_3\mathcal{L}^{-1}\left[\frac{1}{s}\right] = n_3.1 = n_3$$

$$\Rightarrow S_0 = n_3$$

Recall that $A_0 = B_0W_0$ from equation (16)
Also $Z_0 = W_0S_0$ from equation (21)
Hence,

$$\begin{aligned} B_0 &= n_1 \\ W_0 &= n_2 \\ S_0 &= n_3 \\ A_0 &= B_0 W_0 \\ Z_0 &= W_0 S_0 \end{aligned} \tag{31}$$

It follows that

$$\begin{aligned} B_0 &= n_1 \\ W_0 &= n_2 \\ S_0 &= n_3 \\ A_0 &= B_0 W_0 = n_1 n_2 \\ Z_0 &= W_0 S_0 = n_2 n_3 \end{aligned} \tag{32}$$

To find B_1, W_1 and S_1, we substitute equation (32) into equations (28), (29) and (30) respectively and take the inverse Laplace transform. That is, from (28),

$$\mathcal{L}^{-1}\mathcal{L}[B_1] = \mathcal{L}^{-1}\left\{\frac{\phi}{s^{\mu}}\mathcal{L}[B_0] - \frac{\phi}{s^{\mu}q}\mathcal{L}[B_0]^2 - \frac{d_1}{s^{\pi}}\mathcal{L}[A_o]\right\}$$

$$B_1 = \mathcal{L}^{-1}\left\{\frac{\phi}{s^{\mu}}\mathcal{L}[n_1] - \frac{\phi}{s^{\mu}q}\mathcal{L}[n_1]^2 - \frac{d_1}{s^{\pi}}\mathcal{L}[n_1 n_2]\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{\phi}{s^{\mu}} n_1 \mathcal{L}[1] - \frac{\phi}{s^{\mu}q} {n_1}^2 \mathcal{L}[1] - \frac{d_1}{s^{\mu}} n_1 n_2 \mathcal{L}[l]\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{n_1\phi}{s^{\mu}}.\frac{1}{s} - \frac{n^2\phi}{s^{\mu}\,q}.\frac{1}{s} - \frac{n_1 n_2 d_1}{s^{\mu}}.\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{n_1\phi}{s^{\mu+1}} - \frac{{n_1}^2\phi}{qs^{\mu+1}} - \frac{n_1 n_2 d_1}{s^{\mu+1}}\right\}$$

$$= n_1\phi\mathcal{L}^{-1}\left(\frac{1}{s^{\mu+1}}\right) - \frac{{n_1}^2\phi}{q}\mathcal{L}^{-1}\left(\frac{1}{s^{\mu+1}}\right) - n_1 n_2 d_1 \mathcal{L}^{-1}\left(\frac{1}{s^{\mu+1}}\right)$$

From the property of Laplace, $\mathcal{L}[t^{\mu}] = \frac{\mu!}{s^{\mu+1}} \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s^{\mu+1}}\right) = \frac{t^{\mu}}{\mu!}$

Hence,

$$B_1 = \frac{t^{\mu}}{\mu!} n_1\phi - \frac{t^{\mu}}{\mu!}\frac{{n_1}^2\phi}{q} - \frac{t^{\mu}}{\mu!} n_1 n_2 d_1 \tag{33}$$

Also, in (29)

$$\mathcal{L}^{-1}[W_1] = \mathcal{L}^{-1}\left[\frac{\alpha_1}{s^{\mu}}\mathcal{L}[A_0] - \frac{c_1}{s^{\mu}}\mathcal{L}[Z_0] - \frac{d_2}{s^{\mu}}\mathcal{L}[W_0]\right]$$

$$W_1 = \mathcal{L}^{-1}\left[\frac{\alpha_1}{s^{\mu}}\mathcal{L}[n_1 n_2] - \frac{c_1}{s^{\mu}}\mathcal{L}[n_2 n_3] - \frac{d_2}{s^{\mu}}\mathcal{L}[n_2]\right]$$

$$= \mathcal{L}^{-1}\left[\frac{\alpha_1 n_1 n_2}{s^{\mu}}\mathcal{L}[1] - \frac{c_1 n_2 n_3}{s^{\mu}}\mathcal{L}[1] - \frac{d_2 n_2}{s^{\mu}}\mathcal{L}[1]\right]$$

$$= \mathcal{L}^{-1}\left[\frac{\alpha_1 n_1 n_2}{s^{\mu}}.\frac{1}{s} - \frac{c_1 n_2 n_3}{s^{\mu}}.\frac{1}{s} - \frac{d_2 n_2}{s^{\mu}}.\frac{1}{s}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{\alpha_1 n_1 n_2}{s^{\mu+1}} - \frac{c_1 n_2 n_3}{s^{\mu+1}} - \frac{d_2 n_2}{s^{\mu+1}}\right]$$

Hence,

$$W_1 = \frac{t^{\mu}}{\mu!}\alpha_1 n_1 n_2 - \frac{t^{\mu}}{\mu!} c_1 n_2 n_3 - \frac{t^{\mu}}{\mu!} d_2 n_2 \tag{34}$$

Similarly, in (30)

$$\mathcal{L}^{-1}\mathcal{L}[S_1] = \mathcal{L}^{-1}\left[\frac{K}{s^{\mu}} - \frac{c_2}{s^{\mu}}\mathcal{L}[Z_0] - \frac{d_3}{s^{\mu}}\mathcal{L}[S_0]\right]$$

$$S_1 = \mathcal{L}^{-1}\left[\frac{K}{s^{\mu}} - \frac{c_2}{s^{\mu}}\mathcal{L}[n_2 n_3] - \frac{d_3}{s^{\mu}}\mathcal{L}[n_3]\right]$$

$$= \mathcal{L}^{-1}\left[\frac{K}{s^{\mu}} - \frac{c_2 n_2 n_3}{s^{\mu}}\mathcal{L}[1] - \frac{d_3 n_3}{s^{\mu}}\mathcal{L}[1]\right]$$

$$= \mathcal{L}^{-1}\left[\frac{K}{s^{\mu}} - \frac{c_2 n_2 n_3}{s^{\mu+1}} - \frac{d_3 n_3}{s^{\mu+1}}\right]$$

$$S_1 = \frac{t^{\mu}}{\mu!}K - \frac{t^{\mu}}{\mu!} c_2 n_2 n_3 - \frac{t^{\mu}}{\mu!} d_3 n_3 \tag{35}$$

In similar manner, we find B_2, W_2 and S_2 by substituting equation (32) and the values of B_1, W_1, S_1, A_1 and Z_1 into equations, (28), (29) and (30) respectively and take the inverse Laplace transform of the sides.

From (28), i.e, $\mathcal{L}[B_2] = \frac{\phi}{s^{\mu}}\mathcal{L}[B_1] - \frac{\phi}{s^{\mu}q}\mathcal{L}[B_1]^2 - \frac{d_1}{s^{\mu}}\mathcal{L}[A_1]$.

We use B_1 and A_1 to compute B_2 as follows:

$$B_1 = \frac{t^{\mu}}{\mu!} n_1\phi - \frac{t^{\mu}}{\mu!}\frac{{n_1}^2\phi}{q} - \frac{t^{\mu}}{\mu!} n_1 n_2 d_1$$

from (33)

$A_1 = B_0(t)W_1(t) + B_1(t)W_0(t)$ from (20)

But $W_1 = \frac{t^{\mu}}{\mu!}\alpha_1 n_1 n_2 - \frac{t^{\mu}}{\mu!} c_1 n_2 n_3 - \frac{t^{\mu}}{\mu!} d_2 n_2$ from (34)

$$B_0 = n_1, W_0 = n_2$$

Upon substitution,

$$A_1 = n_1\left(\frac{t^{\mu}}{\mu!}\alpha_1 n_1 n_2 - \frac{t^{\mu}}{\mu!} c_1 n_2 n_3 - \frac{t^{\mu}}{\mu!} d_2 n_2\right) + \left(\frac{t^{\mu}}{\mu!} n_1\phi - \frac{t^{\mu}}{\mu!}\frac{{n_1}^2\phi}{q} - \frac{t^{\mu}}{\mu!} n_1 n_2 d_1\right) n_2 \tag{36}$$

We then use (33) and (36) to compute B_2

Hence,

$$B_2 = \frac{t^\mu\phi\left(\frac{t^\mu n_1\phi}{\mu!}-\frac{t^\mu n_1{}^2\phi}{\mu!q}-\frac{t^\mu n_1n_2d_1}{\mu!}\right)}{\mu!} - \frac{t^\mu\phi\left(\left(\frac{t^\mu n_1\phi}{\mu!}-\frac{t^\mu n_1{}^2\phi}{\mu!q}-\frac{t^\mu n_1n_2d_1}{\mu!}\right)^2\right)}{\mu!q} - \frac{t^\mu d_1\left(n_1\left(\frac{t^\mu\alpha_1n_1n_2}{\mu!}-\frac{t^\mu c_1n_2n_3}{\mu!}-\frac{t^\mu d_2n_2}{\mu!}\right)+\left(\frac{t^\mu n_1\phi}{\mu!}-\frac{t^\mu n_1{}^2\phi}{\mu!q}-\frac{t^\mu n_1n_2d_1}{\mu!}\right)n_2\right)}{\mu!} \quad (37)$$

Similarly, from (29), i.e., $\mathcal{L}[W_2] = \frac{\alpha_1}{s^\mu}\,\mathcal{L}[A_1] - \frac{c_1}{s^\mu}\mathcal{L}[Z_1] - \frac{d_2}{s^\mu}\mathcal{L}[W_1]$
We now use the results of A_1, Z_1 and W_1 to compute W_2 in the follow sequence:

$$Z_1 = W_0(t)S_1(t) + W_1(t)S_0(t)$$

Substituting (32), (34) and (35),

$$Z_1 = n_2\left(\frac{t^\mu}{\mu!}K - \frac{t^\mu}{\mu!}c_2n_2n_3 - \frac{t^\mu}{\mu!}d_3n_3\right) + \left(\frac{t^\mu}{\mu!}\alpha_1n_1n_2 - \frac{t^\mu}{\mu!}c_1n_2n_3 - \frac{t^\mu}{\mu!}d_2n_2\right)n_3 \quad (38)$$

We then use (36), (38) and (34) to compute W_2.
Hence,

$$W_2 = \frac{t^\mu\alpha_1\left(n_1\left(\frac{t^\mu\alpha_1n_1n_2}{\mu!}-\frac{t^\mu c_1n_2n_3}{\mu!}-\frac{t^\mu d_2n_2}{\mu!}\right)+\left(\frac{t^\mu n_1\phi}{\mu!}-\frac{t^\mu n_1{}^2\phi}{\mu!q}-\frac{t^\mu n_1n_2d_1}{\mu!}\right)n_2\right)}{\mu!} - \frac{t^\mu c_1\left(n_2\left(\frac{t^\mu K}{\mu!}-\frac{t^\mu c_2n_2n_3}{\mu!}-\frac{t^\mu d_3n_3}{\mu!}\right)+\left(\frac{t^\mu\alpha_1n_1n_2}{\mu!}-\frac{t^\mu c_1n_2n_3}{\mu!}-\frac{t^\mu d_2n_2}{\mu!}\right)n_3\right)}{\mu!} -$$

$$\frac{t^\mu d_2\left(\frac{t^\mu}{\mu!}\alpha_1n_1n_2-\frac{t^\mu c_1n_2n_3}{\mu!}-\frac{t^\mu d_2n_2}{\mu!}\right)}{\mu!} \quad (39)$$

Similarly, from (30), i.e., $\mathcal{L}[S_2] = \frac{K}{s^\mu} - \frac{c_2}{s^\mu}\mathcal{L}[Z_1] - \frac{d_3}{s^\mu}\mathcal{L}[S_1]$
We simply use (38) and (35) to compute S_2.
Hence,

$$S_2 = \frac{t^\mu K}{\mu!} - \frac{t^\mu c_2\left(n_2\left(\frac{t^\mu K}{\mu!}-\frac{t^\mu c_2n_2n_3}{\mu!}-\frac{t^\mu d_3n_3}{\mu!}\right)+\left(\frac{t^\mu\alpha_1n_1n_2}{\mu!}-\frac{t^\mu c_1n_2n_3}{\mu!}-\frac{t^\mu d_2n_2}{\mu!}\right)n_3\right)}{\mu!} - \frac{t^\mu d_3\left(\frac{t^\mu}{\mu!}K-\frac{t^\mu c_2n_2n_3}{\mu!}-\frac{t^\mu d_3n_3}{\mu!}\right)}{\mu!} \quad (40)$$

Next, we compute B_3, W_3 and S_3
From (30), i.e., $\mathcal{L}[B_3] = \frac{\phi}{s^\mu}\mathcal{L}[B_2] - \frac{\phi}{s^\mu q}\mathcal{L}[B_2]^2 - \frac{d_1}{s^\mu}\mathcal{L}[A_2]$.
We now use the results of B_2 and A_2 to compute B_3 in the following sequence: First, we compute A_2.
Recall from (20) that $A_2 = B_0(t)W_2(t) + B_1(t)W_1(t) + B_2(t)W_0(t)$
Substituting (32), (33), (39) and (37) to evaluateA_2, we have

$$A_2 =$$

$$n_1\left(\frac{t^\mu\alpha_1\left(n_1\left(\frac{t^\mu\alpha_1n_1n_2}{\mu!}-\frac{t^\mu c_1n_2n_3}{\mu!}-\frac{t^\mu d_2n_2}{\mu!}\right)+\left(\frac{t^\mu n_1\phi}{\mu!}-\frac{t^\mu n_1{}^2\phi}{\mu!q}-\frac{t^\mu n_1n_2d_1}{\mu!}\right)n_2\right)}{\mu!} - \frac{t^\mu c_1\left(n_2\left(\frac{t^\mu K}{\mu!}-\frac{t^\mu c_2n_2n_3}{\mu!}-\frac{t^\mu d_3n_3}{\mu!}\right)+\left(\frac{t^\mu\alpha_1n_1n_2}{\mu!}-\frac{t^\mu c_1n_2n_3}{\mu!}-\frac{t^\mu d_2n_2}{\mu!}\right)n_3\right)}{\mu!} -\right.$$

$$\left.\frac{t^\mu d_2\left(\frac{t^\mu}{\mu!}\alpha_1n_1n_2-\frac{t^\mu c_1n_2n_3}{\mu!}-\frac{t^\mu d_2n_2}{\mu!}\right)}{\mu!}\right) + \left(\frac{t^\mu n_1\phi}{\mu!}-\frac{t^\mu n_1{}^2\phi}{\mu!q}-\frac{t^\mu n_1n_2d_1}{\mu!}\right)\left(\frac{t^\mu\alpha_1n_1n_2}{\mu!}-\frac{t^\mu c_1n_2n_3}{\mu!}-\frac{t^\mu d_2n_2}{\mu!}\right)\left(\frac{t^\mu\phi\left(\frac{t^\mu n_1\phi}{\mu!}-\frac{t^\mu n_1{}^2\phi}{\mu!q}-\frac{t^\mu n_1n_2d_1}{\mu!}\right)}{\mu!} -\right.$$

$$\left.\frac{t^\mu\phi\left(\left(\frac{t^\mu n_1\phi}{\mu!}-\frac{t^\mu n_1{}^2\phi}{\mu!q}-\frac{t^\mu n_1n_2d_1}{\mu!}\right)^2\right)}{\mu!q} - \frac{t^\mu d_1\left(n_1\left(\frac{t^\mu\alpha_1n_1n_2}{\mu!}-\frac{t^\mu c_1n_2n_3}{\mu!}-\frac{t^\mu d_2n_2}{\mu!}\right)+\left(\frac{t^\mu n_1\phi}{\mu!}-\frac{t^\mu n_1{}^2\phi}{\mu!q}-\frac{t^\mu n_1n_2d_1}{\mu!}\right)n_2\right)}{\mu!}\right)n_2 \quad (41)$$

We then use (37) and (41) to compute B_3
Hence,

$$B_3 =$$

$$\frac{1}{\mu!}\left(t^\mu\phi\left(\frac{t^\mu\phi\left(\frac{t^\mu n_1\phi}{\mu!}-\frac{t^\mu n_1{}^2\phi}{\mu!q}-\frac{t^\mu n_1n_2d_1}{\mu!}\right)}{\mu!} - \frac{t^\mu\phi\left(\left(\frac{t^\mu n_1\phi}{\mu!}-\frac{t^\mu n_1{}^2\phi}{\mu!q}-\frac{t^\mu n_1n_2d_1}{\mu!}\right)^2\right)}{\mu!q} -\right.\right.$$

$$\left.\left.\frac{t^\mu d_1\left(n_1\left(\frac{t^\mu\alpha_1n_1n_2}{\mu!}-\frac{t^\mu c_1n_2n_3}{\mu!}-\frac{t^\mu d_2n_2}{\mu!}\right)+\left(\frac{t^\mu n_1\phi}{\mu!}-\frac{t^\mu n_1{}^2\phi}{\mu!q}-\frac{t^\mu n_1n_2d_1}{\mu!}\right)n_2\right)}{\mu!}\right)\right) -$$

$$\frac{1}{\mu! q}\Bigg(t^{\mu}\phi\Bigg(\frac{t^{\mu}\phi\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)}{\mu!}-\frac{t^{\mu}\phi\left(\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)^{2}\right)}{\mu! q}-$$

$$\frac{t^{\mu}d_1\left(n_1\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)+\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)n_2\right)}{\mu!}\Bigg)^{2}\Bigg)\Bigg)-$$

$$\frac{1}{\mu!}\Bigg(t^{\mu}d_1\Bigg(n_1\Bigg(\frac{t^{\mu}\alpha_1\left(n_1\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)+\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)n_2\right)}{\mu!}-$$

$$\frac{t^{\mu}c_1\left(n_2\left(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2n_2n_3}{\mu!}-\frac{t^{\mu}d_3n_3}{\mu!}\right)+\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)n_3\right)}{\mu!}-\frac{t^{\mu}d_2\left(\frac{t^{\mu}}{\mu!}\alpha_1n_1n_2-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)}{\mu!}\Bigg)+$$

$$\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)+\Bigg(\frac{t^{\mu}\phi\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)}{\mu!}-\frac{t^{\mu}\phi\left(\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)^{2}\right)}{\mu! q}-$$

$$\frac{t^{\mu}d_1\left(n_1\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)+\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)n_2\right)}{\mu!}\Bigg)n_2\Bigg)\Bigg) \qquad (42)$$

from (29), i.e., $\mathcal{L}[W_3]=\frac{\alpha_1}{s^{\mu}}\mathcal{L}[A_2]-\frac{c_1}{s^{\mu}}\mathcal{L}[Z_2]-\frac{d_2}{s^{\mu}}\mathcal{L}[W_2]$

We now use the results of A_2, Z_2 and W_2 to compute W_3 in the follow sequence: First, we compute Z_2. Recall from (25) that $Z_2 = W_0(t)S_2(t)+W_1(t)S_1(t)+W_2(t)S_0(t)$ and evaluating using (32), (34), (35), (39) and (40)

$$Z_2=n_2\Bigg(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2\left(n_2\left(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2n_2n_3}{\mu!}-\frac{t^{\mu}d_3n_3}{\mu!}\right)+\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)n_3\right)}{\mu!}-\frac{t^{\mu}d_3\left(\frac{t^{\mu}}{\mu!}K-\frac{t^{\mu}c_2n_2n_3}{\mu!}-\frac{t^{\mu}d_3n_3}{\mu!}\right)}{\mu!}\Bigg)+\Bigg(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-$$

$$\frac{t^{\mu}d_2n_2}{\mu!}\Bigg)\left(\frac{t^{\mu}}{\mu!}K-\frac{t^{\mu}c_2n_2n_3}{\mu!}-\frac{t^{\mu}d_3n_3}{\mu!}\right)n_3 \qquad (43)$$

We then use (41), (39) and (43) to compute W_3.

Hence,

$$W_3=$$

$$\frac{1}{\mu!}\Bigg(t^{\mu}\alpha_1\Bigg(n_1\Bigg(\frac{t^{\mu}\alpha_1\left(n_1\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)+\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)n_2\right)}{\mu!}-$$

$$\frac{t^{\mu}c_1\left(n_2\left(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2n_2n_3}{\mu!}-\frac{t^{\mu}d_3n_3}{\mu!}\right)+\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)n_3\right)}{\mu!}-\frac{t^{\mu}d_2\left(\frac{t^{\mu}}{\mu!}\alpha_1n_1n_2-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)}{\mu!}\Bigg)+$$

$$\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)+\Bigg(\frac{t^{\mu}\phi\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)}{\mu!}-\frac{t^{\mu}\phi\left(\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)^{2}\right)}{\mu! q}-$$

$$\frac{t^{\mu}d_1\left(n_1\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)+\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)n_2\right)}{\mu!}\Bigg)n_2\Bigg)\Bigg)-$$

$$\frac{t^{\mu}c_1}{\mu!}\Bigg(n_2\Bigg(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2\left(n_2\left(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2n_2n_3}{\mu!}-\frac{t^{\mu}d_3n_3}{\mu!}\right)+\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)n_3\right)}{\mu!}-\frac{t^{\mu}d_3\left(\frac{t^{\mu}}{\mu!}K-\frac{t^{\mu}c_2n_2n_3}{\mu!}-\frac{t^{\mu}d_3n_3}{\mu!}\right)}{\mu!}\Bigg)+\Bigg(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-$$

$$\frac{t^{\mu}d_2n_2}{\mu!}\Bigg)\left(\frac{t^{\mu}}{\mu!}K-\frac{t^{\mu}c_2n_2n_3}{\mu!}-\frac{t^{\mu}d_3n_3}{\mu!}\right)+$$

$$\Bigg(\frac{t^{\mu}\alpha_1\left(n_1\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)+\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}n_1^{2}\phi}{\mu! q}-\frac{t^{\mu}n_1n_2d_1}{\mu!}\right)n_2\right)}{\mu!}-\frac{t^{\mu}c_1\left(n_2\left(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2n_2n_3}{\mu!}-\frac{t^{\mu}d_3n_3}{\mu!}\right)+\left(\frac{t^{\mu}\alpha_1n_1n_2}{\mu!}-\frac{t^{\mu}c_1n_2n_3}{\mu!}-\frac{t^{\mu}d_2n_2}{\mu!}\right)n_3\right)}{\mu!}-$$

$$\frac{t^{\mu}d_2\left(\frac{t^{\mu}}{\mu!}\alpha_1 n_1 n_2-\frac{t^{\mu}c_1 n_2 n_3}{\mu!}-\frac{t^{\mu}d_2 n_2}{\mu!}\right)}{\mu!}\Bigg)n_3\Bigg)-$$

$$\frac{t^{\mu}d_2}{\mu!}\Bigg(\frac{t^{\mu}\alpha_1\left(n_1\left(\frac{t^{\mu}\alpha_1 n_1 n_2}{\mu!}-\frac{t^{\mu}c_1 n_2 n_3}{\mu!}-\frac{t^{\mu}d_2 n_2}{\mu!}\right)+\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}{n_1}^2\phi}{\mu!q}-\frac{t^{\mu}n_1 n_2 d_1}{\mu!}\right)n_2\right)}{\mu!}-\frac{t^{\mu}c_1\left(n_2\left(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2 n_2 n_3}{\mu!}-\frac{t^{\mu}d_3 n_3}{\mu!}\right)+\left(\frac{t^{\mu}\alpha_1 n_1 n_2}{\mu!}-\frac{t^{\mu}c_1 n_2 n_3}{\mu!}-\frac{t^{\mu}d_2 n_2}{\mu!}\right)n_3\right)}{\mu!}-$$

$$\frac{t^{\mu}d_2\left(\frac{t^{\mu}}{\mu!}\alpha_1 n_1 n_2-\frac{t^{\mu}c_1 n_2 n_3}{\mu!}-\frac{t^{\mu}d_2 n_2}{\mu!}\right)}{\mu!}\Bigg) \tag{44}$$

similarly, from (30), i.e.,$\mathcal{L}[S_3]=\frac{K}{s^{\mu}}-\frac{c_2}{s^{\mu}}\mathcal{L}[Z_2]-\frac{d_3}{s^{\mu}}\mathcal{L}[S_2]$

We simply use (43) and (40) to compute S_3.

Hence,

$$\Bigg(n_2\Bigg(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2\left(n_2\left(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2 n_2 n_3}{\mu!}-\frac{t^{\mu}d_3 n_3}{\mu!}\right)+\left(\frac{t^{\mu}\alpha_1 n_1 n_2}{\mu!}-\frac{t^{\mu}c_1 n_2 n_3}{\mu!}-\frac{t^{\mu}d_2 n_2}{\mu!}\right)n_3\right)}{\mu!}-\frac{t^{\mu}d_3\left(\frac{t^{\mu}}{\mu!}K-\frac{t^{\mu}c_2 n_2 n_3}{\mu!}-\frac{t^{\mu}d_3 n_3}{\mu!}\right)}{\mu!}\Bigg)+\Bigg(\frac{t^{\mu}\alpha_1 n_1 n_2}{\mu!}-\frac{t^{\mu}c_1 n_2 n_3}{\mu!}-$$

$$\frac{t^{\mu}d_2 n_2}{\mu!}\Bigg)\left(\frac{t^{\mu}}{\mu!}K-\frac{t^{\mu}c_2 n_2 n_3}{\mu!}-\frac{t^{\mu}d_3 n_3}{\mu!}\right)+$$

$$\Bigg(\frac{t^{\mu}\alpha_1\left(n_1\left(\frac{t^{\mu}\alpha_1 n_1 n_2}{\mu!}-\frac{t^{\mu}c_1 n_2 n_3}{\mu!}-\frac{t^{\mu}d_2 n_2}{\mu!}\right)+\left(\frac{t^{\mu}n_1\phi}{\mu!}-\frac{t^{\mu}{n_1}^2\phi}{\mu!q}-\frac{t^{\mu}n_1 n_2 d_1}{\mu!}\right)n_2\right)}{\mu!}-\frac{t^{\mu}c_1\left(n_2\left(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2 n_2 n_3}{\mu!}-\frac{t^{\mu}d_3 n_3}{\mu!}\right)+\left(\frac{t^{\mu}\alpha_1 n_1 n_2}{\mu!}-\frac{t^{\mu}c_1 n_2 n_3}{\mu!}-\frac{t^{\mu}d_2 n_2}{\mu!}\right)n_3\right)}{\mu!}-$$

$$\frac{t^{\mu}d_2\left(\frac{t^{\mu}}{\mu!}\alpha_1 n_1 n_2-\frac{t^{\mu}c_1 n_2 n_3}{\mu!}-\frac{t^{\mu}d_2 n_2}{\mu!}\right)}{\mu!}\Bigg)n_3\Bigg)-$$

$$\frac{t^{\mu}d_3}{\mu!}\Bigg(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2\left(n_2\left(\frac{t^{\mu}K}{\mu!}-\frac{t^{\mu}c_2 n_2 n_3}{\mu!}-\frac{t^{\mu}d_3 n_3}{\mu!}\right)+\left(\frac{t^{\mu}\alpha_1 n_1 n_2}{\mu!}-\frac{t^{\mu}c_1 n_2 n_3}{\mu!}-\frac{t^{\mu}d_2 n_2}{\mu!}\right)n_3\right)}{\mu!}-\frac{t^{\mu}d_3\left(\frac{t^{\mu}}{\mu!}K-\frac{t^{\mu}c_2 n_2 n_3}{\mu!}-\frac{t^{\mu}d_3 n_3}{\mu!}\right)}{\mu!}\Bigg) \tag{45}$$

2.3. Equilibrium Analysis of the System

Here, we calculate the differential equilibrium points of the system (3), (4) and (5) and check the feasible steady state solutions which will be used in checking the stability of the dynamical system.

(1) The trivial case: i.e., where $\dot{B}=\dot{W}=\dot{S}=0$ and $B=W=S=0$

Substituting into equations (3) to (5), we have

$$\phi B-\frac{\phi B^2}{q}-d_1 BW=0 \tag{46}$$

$$\alpha_1 BW-c_1 WS-d_2 W=0 \tag{47}$$

$$K-c_2 WS-d_3 S=0 \tag{48}$$

i.e., $E_0=(0,0,0)$ is an equilibrium solution

(2) For non-trivial cases:

i. Consider $\dot{B}=0, W=0, S=0, B\neq 0$ and substituting in eqn (46)

$$\phi B-\frac{\phi B^2}{q}-d_1 BW=0$$

$$\phi B-\frac{\phi B^2}{q}-d_1 B(0)=0$$

$$B\left(\phi-\frac{B}{q}\right)=0$$

Since $B\neq 0$, $\left(\phi-\frac{B}{q}\right)=0$

$$\phi=\frac{B}{q}\Longrightarrow B=\phi q \tag{49}$$

Hence, $E_1=(\phi q,0,0)$ is a positive equilibrium where forestry biomass exist, W and S do not exist.

ii. Consider $\dot{S}=0, B=0, W=0, S\neq 0$ and substituting in (48)

$$K-c_2 WS-d_3 S=0$$

$$K-c_2(0)S-d_3 S=0$$

$$K-d_3 S=0$$

$$K=d_3 S\Longrightarrow S=\frac{K}{d_3} \tag{50}$$

Therefore $E_2=(0,0,\frac{K}{d_3})$ is a positive equilibrium where synthetic industry exist, B and W are extinct.

iii. Consider $B\neq 0, W\neq 0, S=0$ and substituting in (47) and (46).

In (47)

$$\alpha_1 BW-c_1 WS-d_2 W=0$$

$$\alpha_1 BW-c_1 W(0)-d_2 W=0$$

$$\alpha_1 BW-d_2 W=0$$

$$W(\alpha_1 B-d_2)=0$$

Since $W\neq 0$, then$\alpha_1 B-d_2=0$

Therefore

$$\alpha_1 B=d_2\Longrightarrow B=\frac{d_2}{\alpha_1} \tag{51}$$

Similarly, in (46)

$$\phi B - \frac{\phi B^2}{q} - d_1 BW = 0$$

$$\phi(\frac{d_2}{\alpha_1}) - \frac{\phi\left(\frac{d_2}{\alpha_1}\right)^2}{q} - d_1(\frac{d_2}{\alpha_1})W = 0$$

$$\frac{\phi d_2}{\alpha_1} - \frac{\phi\frac{d_2{}^2}{\alpha_1{}^2}}{q} - \frac{d_1 d_2 W}{\alpha_1} = 0$$

$$\frac{\phi d_2}{\alpha_1} - \frac{\phi d_2{}^2}{q\alpha_1{}^2} - \frac{d_1 d_2 W}{\alpha_1} = 0$$

Multiply through by α_1

$$\frac{\phi d_2}{1} - \frac{\phi d_2{}^2}{q\alpha_1} - \frac{d_1 d_2 W}{1} = 0$$

$$\frac{\phi q\alpha_1 d_2 - \phi d_2{}^2}{q\alpha_1} = d_1 d_2 W$$

$$\frac{\frac{\phi q\alpha_1 d_2 - \phi d_2{}^2}{q\alpha_1}}{d_1 d_2} = W$$

Therefore,

$$W = \frac{\phi q\alpha_1 d_2 - \phi d_2{}^2}{q\alpha_1 d_1 d_2} \tag{52}$$

Hence, $E_3 = \left(\frac{d_2}{\alpha_1}, \frac{\phi q\alpha_1 d_2 - \phi d_2{}^2}{q\alpha_1 d_1 d_2}, 0\right)$ is a positive equilibrium where S does not exist

2.4. Stability Analysis

Considering the three interacting functions

$$F_1(B, W, S) = \phi B - \frac{\phi B^2}{q} - d_1 BW \tag{53}$$

$$F_2(B, W, S) = \alpha_1 BW - c_1 WS - d_2 W \tag{54}$$

$$F_3(B, W, S) = K - c_2 WS - d_3 S \tag{55}$$

To check the various steady state solutions, we have to obtain the Jacobian matrix corresponding to the systems.

Differentiating (53) with respect to B, W and S respectively, we have

$$J_{11} = \frac{\partial F_1}{\partial B} = \phi - \frac{2\phi B}{q} - d_1 W$$

$$J_{12} = \frac{\partial F_1}{\partial W} = -d_1 B \tag{56}$$

$$J_{13} = \frac{\partial F_1}{\partial W} = 0$$

Differentiating (3.54) with respect to B, W and S respectively, we have

$$J_{21} = \frac{\partial F_2}{\partial B} = \alpha_1 W$$

$$J_{22} = \frac{\partial F_2}{\partial W} = \alpha_1 B - c_1 S - d_2 \tag{57}$$

$$J_{23} = \frac{\partial F_2}{\partial S} = -c_1 W$$

Differentiating (55) with respect to B, W and S respectively, we have

$$J_{31} = \frac{\partial F_3}{\partial B} = 0$$

$$J_{32} = \frac{\partial F_3}{\partial W} = -c_2 S \tag{58}$$

$$J_{33} = \frac{\partial F_3}{\partial S} = -c_2 W - d_3$$

Setting up the Jacobian matrix, we have

$$J = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}$$

$$J = \begin{pmatrix} \phi - \frac{2\phi B}{q} - d_1 W & -d_1 B & 0 \\ \alpha_1 W & \alpha_1 B - c_1 S - d_2 & -c_1 W \\ 0 & -c_2 S & -c_2 W - d_3 \end{pmatrix} \tag{59}$$

With the characteristics equation defined as $|J - \lambda I| = 0$. Where I is the unit matrix and λ is the Eigen value. We then evaluate the Jacobian matrix at the equilibrium points as follows:

$$J_{/0,0,0} = \begin{pmatrix} \phi & 0 & 0 \\ 0 & -d_2 & 0 \\ 0 & 0 & -d_3 \end{pmatrix}$$

The characteristic equation corresponding to the matrix is given as

$$(\phi - \lambda)(-d_2 - \lambda)(-d_3 - \lambda) = 0$$

On solving, we have $\lambda_1 = \phi$, $\lambda_2 = -d_2$, $\lambda_3 = -d_3$. The Eigen value has both positive and negative values. Hence, unstable at the fix points.

Similarly,

$$J_{/\phi q,0,0} = \begin{pmatrix} \phi - \frac{2\phi(\phi q)}{q} & -d_1(\phi q) & 0 \\ 0 & \alpha_1(\phi q) - d_2 & 0 \\ 0 & 0 & -d_3 \end{pmatrix}$$

The characteristic equation corresponding to the matrix is given as

$$(\phi - 2\phi^2 - \lambda)(\alpha_1\phi q - d_2 - \lambda)(-d_3 - \lambda) = 0$$

On solving, we have $\lambda_1 = \phi - 2\phi^2, \lambda_2 = \alpha_1\phi q - d_2, \lambda_3 = -d_3$ implying that the Eigen values are both negative and positive. λ_2 is negative when $d_2 > \alpha_1\phi q$, and since all the values of λ will be less than zero, we therefore conclude that the system becomes stable at the evaluated fix points.

Also,

$$J/_{0,0,\frac{K}{d_3}} = \begin{pmatrix} \phi & 0 & 0 \\ 0 & -c_1\left(\frac{K}{d_3}\right) - d_2 & 0 \\ 0 & -c_2\left(\frac{K}{d_3}\right) & -d_3 \end{pmatrix}$$

The characteristic equation corresponding to the matrix is given as

$$(\phi - \lambda)\left(\frac{-c_1 K}{d_3} - d_2 - \lambda\right)(-d_3 - \lambda) = 0$$

On solving, we have $\lambda_1 = \phi$, $\lambda_2 = \frac{-c_1 K}{d_3} - d_2$, $\lambda_3 = -d_3$.The Eigen values are both positive and negative. The system is stable if ϕ is negative.

Thus, the proposed Laplace Decomposition Method for the analysis in section (3), provides us with the following series solutions:

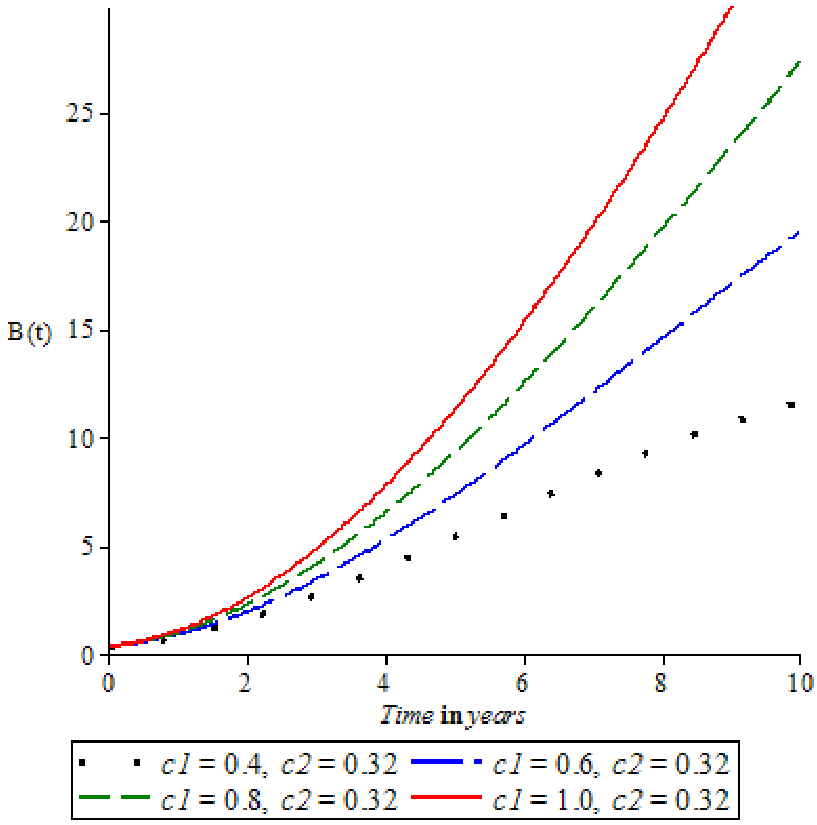

Figure 1. *Plot of numerical solution of forest biomass B(t) corresponding to different time (t) in years.*

3. Results and Discussions

3.1. Forest Biomass Density

$$B(t) = B_0 + \frac{t^{\mu} n_1 \phi}{\mu!} - \frac{t^{\mu} {n_1}^2 \phi}{\mu! q} - \frac{t^{\mu} n_1 n_2 d_1}{\mu!} + \frac{t^{\mu} \phi B_1}{\mu!} - \frac{t^{\mu} \phi {B_1}^2}{\mu! q} - \frac{t^{\mu} \phi d_1 A_1}{\mu!} \quad (60)$$

For $c_1 = 0.4, c_2 = 0.32$

$$B(t) = B_r = 0.35 + \frac{t^{\mu} n_1 \phi}{\mu!} - \frac{t^{\mu} {n_1}^2 \phi}{\mu!\, q} - \frac{t^{\mu} n_1 n_2 d_1}{\mu!} + \frac{t^{\mu} \phi B_1}{\mu!} - \frac{t^{\mu} \phi {B_1}^2}{\mu!\, q} - \frac{t^{\mu} \phi d_1 A_1}{\mu!}$$

$$B_r = 0.35 + 0.2819250000t + 0.2036926500\, t^2 - 0.01192225584\, t^3$$

For $c_1 = 0.6, c_2 = 0.32$

$$B(t) = B_u = 0.35 + \frac{t^{\mu} n_1 \phi}{\mu!} - \frac{t^{\mu} {n_1}^2 \phi}{\mu!\, q} - \frac{t^{\mu} n_1 n_2 d_1}{\mu!} + \frac{t^{\mu} \phi B_1}{\mu!} - \frac{t^{\mu} \phi {B_1}^2}{\mu!\, q} - \frac{t^{\mu} \phi d_1 A_1}{\mu!}$$

$$B_u = 0.35 + 0.2819250000t + 0.2823376500\, t^2 - 0.01192225584\, t^3$$

For $c_1 = 0.8,\ c_2 = 0.32$

$$B(t) = B_v = 0.35 + \frac{t^{\mu} n_1 \phi}{\mu!} - \frac{t^{\mu} {n_1}^2 \phi}{\mu!\, q} - \frac{t^{\mu} n_1 n_2 d_1}{\mu!} + \frac{t^{\mu} \phi B_1}{\mu!} - \frac{t^{\mu} \phi {B_1}^2}{\mu!\, q} - \frac{t^{\mu} \phi d_1 A_1}{\mu!}$$

$$B_v = 0.35 + 0.2819250000t + 0.3609826500\, t^2 - 0.01192225584\, t^3$$

For $c_1 = 1.0, c_2 = 0.32$

$$B(t) = B_p = 0.35 + \frac{t^{\mu} n_1 \phi}{\mu!} - \frac{t^{\mu} {n_1}^2 \phi}{\mu!\, q} - \frac{t^{\mu} n_1 n_2 d_1}{\mu!} + \frac{t^{\mu} \phi B_1}{\mu!} - \frac{t^{\mu} \phi {B_1}^2}{\mu!\, q} - \frac{t^{\mu} \phi d_1 A_1}{\mu!}$$

$$B_p = 0.35 + 0.2819250000t + 0.4396276500\, t^2 - 0.01192225584\, t^3$$

Wood based industries;

$$W(t) = W_0 + \frac{t^{\mu} n_1 n_2 \alpha_1}{\mu!} - \frac{t^{\mu} c_1 n_2 n_3}{\mu!} - \frac{t^{\mu} d_2 n_2}{\mu!} + \frac{t^{\mu} \alpha_1 A_1}{\mu!} - \frac{t^{\mu} c_1 Z_1}{\mu!} - \frac{t^{\mu} d_2 W_1}{\mu!} + \frac{t^{\mu} \alpha_1 A_2}{\mu!} - \frac{t^{\mu} c_1 Z_2}{\mu!} - \frac{t^{\mu} d_2 W_2}{\mu!} \quad (61)$$

For $c_1 = 0.4, c_2 = 0.32$

$$W(t) = W_r = 1.07 + \frac{t^{\mu} n_1 n_2 \alpha_1}{\mu!} - \frac{t^{\mu} c_1 n_2 n_3}{\mu!} - \frac{t^{\mu} d_2 n_2}{\mu!} + \frac{t^{\mu} \alpha_1 A_1}{\mu!} - \frac{t^{\mu} c_1 Z_1}{\mu!} - \frac{t^{\mu} d_2 W_1}{\mu!} + \frac{t^{\mu} \alpha_1 A_2}{\mu!} - \frac{t^{\mu} c_1 Z_2}{\mu!} - \frac{t^{\mu} d_2 W_2}{\mu!}$$

$$W_r = 1.07 + 0.18190t + 0.454637650t^2 + 3.8\big(t\,(0.4283564705t^2 - 0.01275681375\, t^3)\big) - 0.4\big(t\,(5.35t - 2.424861466\, t^2)\big) - 0.2091333190t^3$$

For $c_1 = 0.4, c_2 = 0.52$

$$W(t) = W_u = 1.07 + \frac{t^{\mu} n_1 n_2 \alpha_1}{\mu!} - \frac{t^{\mu} c_1 n_2 n_3}{\mu!} - \frac{t^{\mu} d_2 n_2}{\mu!} + \frac{t^{\mu} \alpha_1 A_1}{\mu!} - \frac{t^{\mu} c_1 Z_1}{\mu!} - \frac{t^{\mu} d_2 W_1}{\mu!} + \frac{t^{\mu} \alpha_1 A_2}{\mu!} - \frac{t^{\mu} c_1 Z_2}{\mu!} - \frac{t^{\mu} d_2 W_2}{\mu!}$$

$$W_u = 1.07 + 0.18190t + 0.614923650\, t^2 + 3.8\, \big(t(0.4844565705\, t^2 - 0.01275681375\, t^3)\big) - 0.4\big(t(5.35t - 1.823124924\, t^2)\big) - 0.2828648790t^3$$

For $c_1 = 0.4, c_2 = 0.72$

$$W(t) = W_v = 1.07 + \frac{t^{\mu} n_1 n_2 \alpha_1}{\mu!} - \frac{t^{\mu} c_1 n_2 n_3}{\mu!} - \frac{t^{\mu} d_2 n_2}{\mu!} + \frac{t^{\mu} \alpha_1 A_1}{\mu!} - \frac{t^{\mu} c_1 Z_1}{\mu!} - \frac{t^{\mu} d_2 W_1}{\mu!} + \frac{t^{\mu} \alpha_1 A_2}{\mu!} - \frac{t^{\mu} c_1 Z_2}{\mu!} - \frac{t^{\mu} d_2 W_2}{\mu!}$$

$$W_v = 1.07 + 0.18190t + 0.775209650t^2 + 3.8\, \big(t(0.5405566705t^2 - 0.01275681375t^3)\big) - 0.4\big(t(5.35t - 1.049882362t^2)\big) - 0.3565964390t^3$$

For $c_1 = 0.4, c_2 = 0.92$

$$W(t) = W_p = 1.07 + \frac{t^{\mu} n_1 n_2 \alpha_1}{\mu!} - \frac{t^{\mu} c_1 n_2 n_3}{\mu!} - \frac{t^{\mu} d_2 n_2}{\mu!} + \frac{t^{\mu} \alpha_1 A_1}{\mu!} - \frac{t^{\mu} c_1 Z_1}{\mu!} - \frac{t^{\mu} d_2 W_1}{\mu!} + \frac{t^{\mu} \alpha_1 A_2}{\mu!} - \frac{t^{\mu} c_1 Z_2}{\mu!} - \frac{t^{\mu} d_2 W_2}{\mu!}$$

$$W_p = 1.07 + 0.18190t + 0.935495650t^2 + 3.8\, \big(t(0.5966567705\, t^2 - 0.01275681375\, t^3)\big) - 0.4\big(t(5.35t - 0.1051337800\, t^2)\big) - 0.4303279990t^3$$

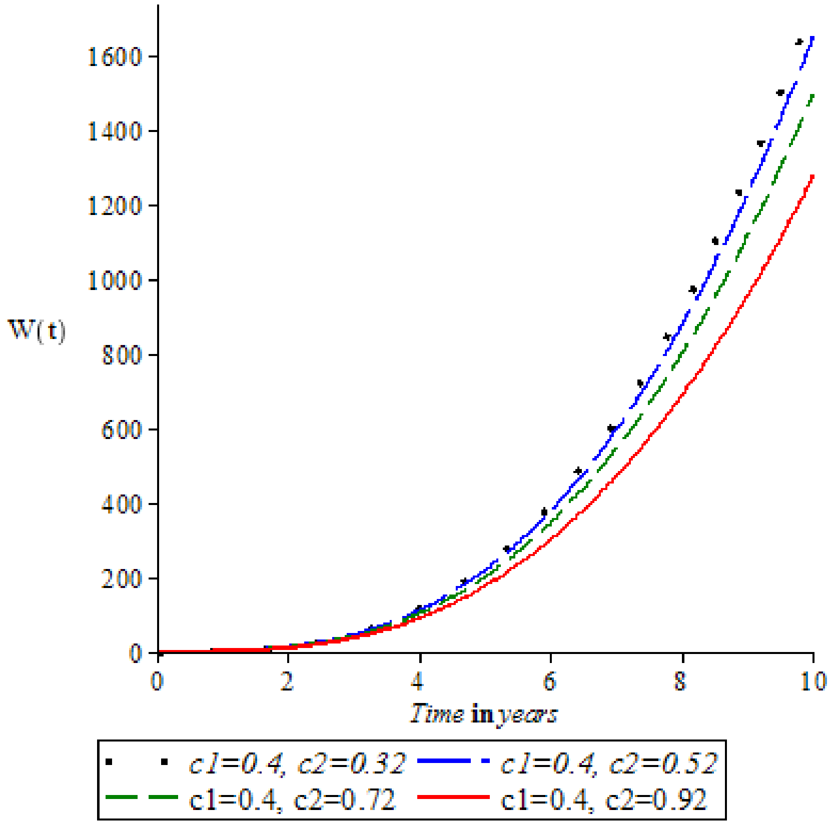

Figure 2. *Plot of numerical solution of wood based industries W(t) corresponding to different time in years.*

For the synthetic industries;

$$S(t) = S_0 + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2n_2n_2}{\mu!} - \frac{t^{\mu}d_3n_3}{\mu!} + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2Z_1}{\mu!} - \frac{t^{\mu}d_3S_1}{\mu!} + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2Z_2}{\mu!} - \frac{t^{\mu}d_3S_2}{\mu!} \tag{62}$$

$$K = 5$$

$$S(t) = S_r = 1.75 + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2n_2n_2}{\mu!} - \frac{t^{\mu}d_3n_3}{\mu!} + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2Z_1}{\mu!} - \frac{t^{\mu}d_3S_1}{\mu!} + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2Z_2}{\mu!} - \frac{t^{\mu}d_3S_2}{\mu!}$$

$$S_r = 1.75 + 10.564800\,t - 1.724288120\,t^2 + 0.92\left(t(5.35\,t - 0.1051337800\,t^2)\right) - 1.55\left(t(5t - 1.724288120\,t^2)\right)$$

$$K = 5.2$$

$$S(t) = S_u = 1.75 + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2n_2n_2}{\mu!} - \frac{t^{\mu}d_3n_3}{\mu!} + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2Z_1}{\mu!} - \frac{t^{\mu}d_3S_1}{\mu!} + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2Z_2}{\mu!} - \frac{t^{\mu}d_3S_2}{\mu!}$$

$$S_u = 1.75 + 11.164800\,t - 2.231168120\,t^2 + 0.92\left(t(5.564\,t - 0.7609153800\,t^2)\right) - 1.55\left(t(5.2t - 2.231168120\,t^2)\right)$$

$$K = 5.4$$

$$S(t) = S_v = 1.75 + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2n_2n_2}{\mu!} - \frac{t^{\mu}d_3n_3}{\mu!} + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2Z_1}{\mu!} - \frac{t^{\mu}d_3S_1}{\mu!} + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2Z_2}{\mu!} - \frac{t^{\mu}d_3S_2}{\mu!}$$

$$S_v = 1.75 + 11.764800\,t - 2.738048120\,t^2 + 0.92\left(t(5.778t - 1.416696980\,t^2)\right) - 1.55\left(t(5.4\,t - 2.738048120\,t^2)\right)$$

$$K = 5.6$$

$$S(t) = S_p = 1.75 + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2n_2n_2}{\mu!} - \frac{t^{\mu}d_3n_3}{\mu!} + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2Z_1}{\mu!} - \frac{t^{\mu}d_3S_1}{\mu!} + \frac{t^{\mu}K}{\mu!} - \frac{t^{\mu}c_2Z_2}{\mu!} - \frac{t^{\mu}d_3S_2}{\mu!}$$

$$S_p = 1.75 + 12.364800\,t - 3.244928120\,t^2 + 0.92\left(t(5.992\,t - 2.072478580\,t^2)\right) - 1.55\left(t(5.6\,t - 3.244928120\,t^2)\right)$$

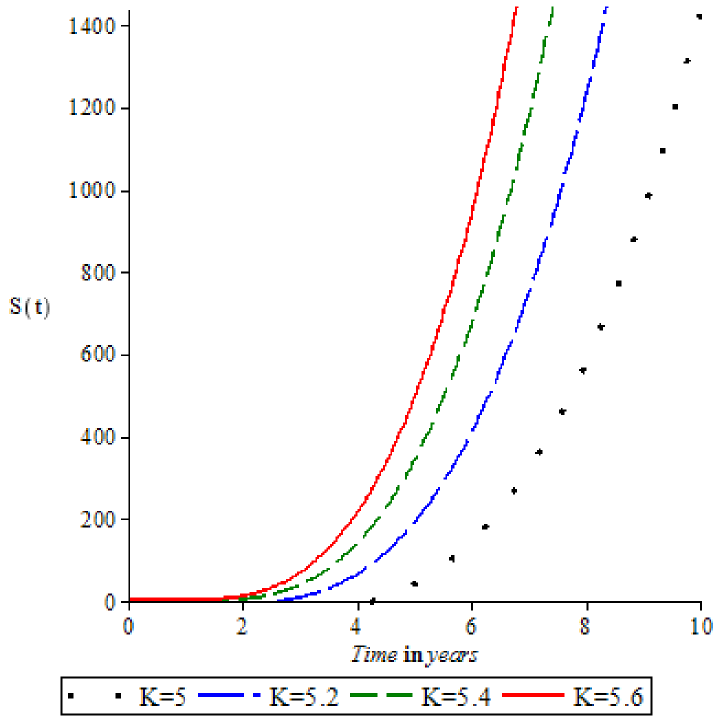

Figure 3. *Plot of numerical solution of synthetic industries S(t) corresponding to different time in years.*

3.2. Findings

The behavior of the different population under investigation at different model parameter values are implemented using Maple18 computer software, and are presented graphically.

In Figure 1, it is observed that as we increase the competitive effect of forest biomass c_1 on wood industries from 0.4 to 1.0 and keeping the competitive effect of wood based industries c_2, on forest biomass constant at 0.32, the solution trajectories of the forest biomass B(t) increases rapidly within a short period of time. This implies that the time (t) to achieve a geometrical growth of the forest population B(t) reduces, and the forest population will explode quickly. This can be achieved by encouraging afforestation and providing sufficient synthetics as alternatives to cushion the effect of wood based industries on the forest biomass.

Similarly, in Figure 2, it is observed that when c_1 is kept constant at 0.4, and the competitive effect of wood based industries c_2, on forest biomass is increased from 0.32 to 0.92, the solution trajectories of the wood industries W(t) decreases rapidly as time (t) progresses. This indicates that the population of wood industries will have less effect on the density of forest biomass and will soon go into extinction. This can be achieved by a drop in demand for wood market, due to awareness of synthetic alternative, increase in prices of timber and government regulations on deforestation.

Also, Figure 3 shows that as the parameter value for the amount of synthetic materials K, supplied to the synthetic industries increases from 5 to 5.6, the solution trajectories of the synthetic industries S(t) increases geometrically within a short period of time. This indicates that the synthetic industries will explode as sufficient synthetic materials are supplied into the industries.

4. Conclusions

In this research work, the process for modeling and numerical analysis of forestry conservation was carried out using the Laplace Adomian Decomposition Method for its analysis. Three population classes were presented in the model, namely; the density of forest biomass, the density of wood based industries, and population of synthetic industries. The LADM which is an applicable algorithm for solving systems of non-linear differential equations was employed. It is obvious that the results obtained using LADM is very reliable. It equally indicates that the method used can predict accurately the behavior of the various populations under investigation. This confirms that the forestry biomass may be conserved from being depleted by the wood based industries if synthetics are promoted as useful alternatives to wood, and regulations on deforestation for timber product (wood) is put in place by relevant government agencies.

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